

# Birational invariance of higher Amitsur groups (joint with Yuri Tschinkel and Zhijie Zhang)

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## 1. Equivariant birational geometry

$k$  field char 0

$G$  finite (abstract) group

$X, Y$   $G$ -varieties

- $X, Y$   $G$ -birationally equivalent if  $\exists X \rightsquigarrow Y$   $G$ -equiv bir equivalence  
stably " " if  $\exists m, n \geq 0$   $X \times \mathbb{P}_k^m$  and  $Y \times \mathbb{P}_k^n$  are  $G$ -bir equiv  
( $G$  acts trivially on  $\mathbb{P}_k^m, \mathbb{P}_k^n$ )
- $X$  is (stably)  $G$ -linearizable if it is (stably)  $G$ -bir equiv to  $\mathbb{P}(V)$   $V$   $G$ -rep.
- $X$  is  $G$ -unirational if  $\exists \mathbb{P}(V) \dashrightarrow X$   $G$ -equiv dom n't l map,  $V$  linear  $G$ -rep

Open problem: Classify  $G$ -varieties up to (stable)  $G$ -birational equiv.

Subproblem: Classify conjugacy classes of finite subgroups of the Cremona group  $Cr_d(k) := Bir(\mathbb{P}_k^d)$ .

## 2. Higher Amitsur groups

$S$   $k$ -split  $G$ -torsor ( $S \subseteq \mathbb{G}_m^n$ ,  $G \curvearrowright S$  by group act  $G \rightarrow \text{Aut}(S) = GL_n(k)$ )

Ex:  $S = \mathbb{G}_m$  trivial  $G$ -action

$P_{\text{ex}} X$   $\mathbb{Z}$ -free of finite rank,  $S = T_{NS}$  Néron-Severi torsor

$$X_{\text{ex}}^*(S) = P_{\text{ex}}(X) = NS(X).$$

$$\uparrow \uparrow$$

$$H_{\text{ét}}^*(\mathbb{G}_m, S)$$

Assume  $X$  smooth and  $k[X]^G = k^X$  (e.g.  $X$  smooth proj geom conn)

$$E_2^{pq} := H^p(G, H^q(X, S)) \Rightarrow H_G^{pq}(X, S) \quad \left| \begin{array}{l} H^*(G, -) \text{ group coh } (-)^G \\ H^*(X, -) \text{ étale coh } \Gamma(X, -) \\ H_G^*(X, -) \text{ } G\text{-equiv étale coh } \Gamma(X, -)^G \end{array} \right.$$

$$\text{Def: } \forall m \geq 2 \quad Am^m(G \curvearrowright X, S) := \text{Im}(d_2^{m-2, 1}) \subset H^m(G, S(k))$$

Remk: If  $X(k)^G \neq \emptyset$  or  $X$   $G$ -unirad  $\Rightarrow Am^m = 0 \quad \forall m \geq 2$ .

$$S = \mathbb{G}_m \quad Am^m(G \curvearrowright X) := Am^m(G \curvearrowright X, \mathbb{G}_m).$$

$$m=2 \quad Am^2(G \curvearrowright X) = \text{Im}(P_{\text{ex}}(X)^G \rightarrow H^2(G, k^X))$$

Low degree exact seq

$$G \curvearrowright d^3 \quad \dots 2m-1 \times 1$$

$$m=2$$

Low degree exact seq

$$0 \rightarrow H^1(G, k^*) \rightarrow \text{Pic}_G(X) \rightarrow \text{Pic}(X) \xrightarrow{d_2^3} H^2(G, k^*)$$

$\Rightarrow \text{Am}^2$  obstruction  $\in G$ -lin of l.b.  $G$ -linearized l.b.  $(= \text{Pic}([X/G]))$

Prop:  $\text{Am}^2$  is a stable  $G$ -lin inv. of smooth proj  $k$ -var.

Proof:  $Z = \coprod Z_i \subset X$   $G$ -stable smooth closed codim  $\geq 2$

$$E \hookrightarrow \tilde{X} = \text{Bl}_Z X \quad E = \coprod E_i$$

$$\downarrow \quad \downarrow$$

$$Z \hookrightarrow X$$

$$\text{Pic}(\tilde{X}) = \text{Pic}(X) \oplus P, \quad P = \bigoplus_i \mathbb{Z}[\mathcal{O}(E_i)] \quad \text{permutation } G\text{-lattice}$$

$$\begin{array}{ccc} \text{Pic}(\tilde{X})^G & \xrightarrow{\quad} & H^2(G, k^*) \\ \downarrow \text{([O(E)])} & & \downarrow \\ \text{Pic}(X)^G \oplus \mathbb{Z}[\mathcal{O}(E)] & \xrightarrow{\quad} & 0 \end{array}$$

$$\Rightarrow \text{Am}^2(G \curvearrowright \tilde{X}) = \text{Am}^2(G \curvearrowright X).$$

$$\text{Similarly } \text{Am}^2(G \curvearrowright X \times \mathbb{P}^n) = \text{Am}^2(G \curvearrowright X)$$

Weak factorization  $\Rightarrow \text{Am}^2$  st  $G$ -lin inv.

$$m=3 \quad \text{Am}^3(G \curvearrowright X) = \text{Im} (H^1(G, \text{Pic} X) \rightarrow H^3(G, k^*))$$

Prop:  $\text{Am}^3$  st.  $G$ -lin inv. of sm proj  $G$ -var.

$$\text{Proof: } H^1(G, \text{Pic} \tilde{X}) = H^1(G, \text{Pic} X) \oplus H^1(G, P) = H^1(H, \mathbb{Z}) = 0 \quad (H = \text{Stab}(Z, 1))$$

W.F  $\Rightarrow \square$ .

$$m=4$$

Kresch-Tschinkel 2026:  $\text{Am}^4$  st.  $G$ -lin inv of  $k$ -rational smooth proj var

Proof uses blowup formula in motivic cohomology.  $(\mathbb{E}_m = \mathbb{Z}(1)[1])$

$$\text{Rmk: } H^1(G, P) = 0 \quad H^i(G, P) \neq 0 \text{ in gen } \forall i > 1$$

### 3. Main Theorem

Theorem<sup>1</sup> (S.-Tschinkel - Zhang 2026):  $\forall k$ -split  $G$ -forms  $S, \forall m \geq 2$   
 $\text{Am}^m(-, S)$  is a stable  $G$ -lin inv of smooth proj  $G$ -var.

For the proof, we assume  $S = \mathbb{G}_m, X/k$  geom conn.

$$(*) \quad 1 \rightarrow k^* \rightarrow k(X)^* \xrightarrow{\text{div}} \text{Div } X \rightarrow \text{Pic } X \rightarrow 0 \quad G\text{-equiv}$$

$$\alpha(G \curvearrowright X) \in \text{Ext}_G^2(\text{Pic } X, k^*) \text{ ext class.}$$

$$\alpha(G \curvearrowright X) \in \text{Ext}_G^2(\text{Rc } X, k^*) \text{ ext class.}$$

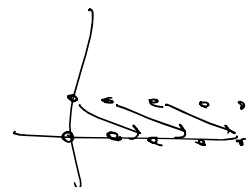
Theorem 2 (S.-Tschinkel-Zhang '26):  $d_2^{n-2} = (-) \circ \alpha(G \curvearrowright X): H^*(G, \text{Rc } X) \rightarrow H^*(G, k^*)$

Proof: Change-of-site map  $X_{\text{ét}} \rightarrow X_{\text{Zar}}$

$$E_2^{pq} := H^p(G, H^q(X, \mathbb{G}_m)) \Rightarrow H_G^{p+q}(X, \mathbb{G}_m)$$

$$\uparrow$$

$$(E_{\text{Zar}})_2^{pq} := H^p(G, H^q_{\text{Zar}}(X, \mathbb{G}_m)) \Rightarrow H_{G, \text{Zar}}^{p+q}(X, \mathbb{G}_m)$$



$$H^1(X, \mathbb{G}_m) = H^1_{\text{Zar}}(X, \mathbb{G}_m) = \text{Rc}(X). \text{ + naturality}$$

$$\Rightarrow \text{ETS} \quad (d_{\text{Zar}})_2^{n-2} = (-) \circ \alpha(G \curvearrowright X).$$

$$1 \rightarrow \mathbb{G}_m \rightarrow j_* \mathbb{G}_m, k(X) \xrightarrow{\text{div}} \underline{\text{Div}}_X \rightarrow 0$$

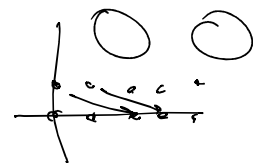
$G$ -equiv exact seq of Zariski sheaves

Flesque resolution of  $\mathbb{G}_m$  on  $X_{\text{Zar}}$

$$\text{RT}_{\text{Zar}}(X, \mathbb{G}_m) \simeq [k(X)^* \xrightarrow{\text{div}} \underline{\text{Div}}_X]$$

$$j: \text{Spec } k(X) \hookrightarrow X$$

$$\underline{\text{Div}}_X = \bigoplus_{x \in X^{(1)}} (i_x)_* \mathbb{Z} \quad i_x: \text{Spec } k(x) \hookrightarrow X.$$



Homological algebra  $\Rightarrow (d_{\text{Zar}})_2^{n-2}$  are given by stable connecting maps for seq (\*), i.e. cup product by ext class  $\alpha(G \curvearrowright X)$ .

Proof of Thm 1

$$\begin{array}{ccc} E & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow \pi \end{array}$$

$$(\pi^*, \varepsilon): \text{Rc}(X) \oplus P \xrightarrow{\sim} \text{Rc } \tilde{X} \quad \varepsilon: [\mathcal{O}(E)] \mapsto [\mathcal{O}(E_0)]$$

$$\begin{array}{ccc} \tilde{Z} \hookrightarrow \tilde{X} & \xrightarrow{(\pi^*, \varepsilon)} & \text{Ext}_G^2(\text{Rc } \tilde{X}, k^*) \xrightarrow{\sim} \text{Ext}_G^2(\text{Rc } X, k^*) \oplus \text{Ext}_G^2(P, k^*) \\ & & \alpha(G \curvearrowright \tilde{X}) \xrightarrow{?} (\alpha(G \curvearrowright X), 0) \end{array}$$

$$1 \rightarrow k^* \rightarrow k(X)^* \rightarrow \underline{\text{Div}} \tilde{X} \rightarrow \text{Rc } \tilde{X} \rightarrow 0$$

$$\begin{array}{ccccccc} & & \pi^* \uparrow & & \pi^* \uparrow & & \pi^* \uparrow \\ 1 \rightarrow & k^* & \rightarrow & k(X)^* & \rightarrow & \underline{\text{Div}} X & \rightarrow \text{Rc } X \rightarrow 0 \end{array}$$

$$1 \rightarrow k^* \rightarrow k(\tilde{X})^* \rightarrow \underline{\text{Div}} \tilde{X} \rightarrow \text{Rc } \tilde{X} \rightarrow 0$$

$$\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \uparrow \\ 1 \rightarrow & k^* & \rightarrow & k(\tilde{X}) & \rightarrow & M & \xrightarrow{\quad} P \rightarrow 0 \end{array}$$

$$H^{m-2}(G, \mathbb{Q}) \simeq H^{m-2}(G, \mathbb{Q}) \oplus H^{m-2}(G, P)$$

$$H^{m-2}(G, \text{Pic } \tilde{X}) = H^{m-2}(G, \text{Pic } X) \oplus H^{m-2}(G, \mathbb{P})$$

$$\downarrow d_2^{m-2,1} = (-) \cup \alpha(G \curvearrowright \tilde{X}) \quad \downarrow d_2^{m-2,1} = (-) \cup (\alpha(G \curvearrowright X), 0)$$

$$H^m(G, \mathbb{R}^*) = H^m(G, \mathbb{R}^*)$$

$\Rightarrow Am^m(G \curvearrowright \tilde{X}) = Am^m(G \curvearrowright X)$ .  
 Similarly for  $X \hookrightarrow X \times \mathbb{P}^m$ . W.F. D.

#### 4. Universal torsors

$$H^1(X, T_{NS})^G$$

S.S. for  $S = T_{NS}$   $d_2^{0,1}: \text{End}(\text{Pic } X) \xrightarrow{\hookrightarrow} H^2(G, T_{NS}(\mathbb{R}))$

$$\cup$$

$$\text{id} \longmapsto \beta(G \curvearrowright X).$$

$\beta$  obstructs lifting  $G$ -action to universal torsor  $T_X \rightarrow X$   
 $G$ -uniseriality

Kresch-Totaro 2006:  $\beta(G \curvearrowright X) = 0$

Then  $Am^2(G \curvearrowright X) = 0$ . If  $G \curvearrowright X$  toric  $\Rightarrow Am^m(G \curvearrowright X) = 0 \quad \forall m \geq 2$

Theorem 3 (S.T.Z '26):  $\forall X$  ( $\text{Pic } X \otimes \mathbb{Z}$ -free finite rank)

$\beta(G \curvearrowright X) = 0 \Rightarrow Am^m(G \curvearrowright X, S) = 0 \quad \forall m \geq 2$ .

Proof: homological alg  $\Rightarrow \beta(G \curvearrowright X) = -\alpha(G \curvearrowright X)$

$$H^2(G, T_{NS}(\mathbb{R})) = \text{Ext}^2(\text{Pic } X, \mathbb{R}^*)$$

$$X = \{w^2 = x_1^4 + x_2^4 + x_3^4\} \subset \mathbb{P}(2, 1, 1) \quad / \mathbb{C} \quad G = \mathbb{Z}/8 \times \mathbb{Z}/2$$

$$\beta \neq 0. \quad Am^m = 0 \quad \forall m \geq 2. \quad \sigma \quad \tau \quad \tau \sigma \tau^{-1} = \sigma^5$$